উচ্চমাধ্যমিক গণিত – ২০২২

নির্বাচিত প্রশ্নাবলী

1. যদি $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$তহয় তাহলে প্রমাণ কর $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

ধরি, $x=\sin\theta;\ y=\sin\phi$

$\therefore\ \cos\theta+\cos\phi=a(\sin\theta-\sin\phi)$

$\therefore\ 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=a.2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$

$\therefore\ \cot\frac{\theta-\phi}{2}=a$

$\therefore\ \theta-\phi=2\cot^{-1}a\ \Rightarrow\sin^{-1}x-\sin^{-1}y=2\cot^{-1}x$

$\therefore\ \frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$

2. যদি $f(x)=(x+1)^x+x^{x+1}$তয় তাহলে $f'(1)$ এর মান নির্ণয় কর :

$f(x)=(x+1)^x+x^{x+1}$

$=\ e^{\ln(x+1)^x}+e^{\ln x^{x+1}}$

$f(x)=e^{x\ln(x+1)}+e^{(x+1)\ln x}$

$\therefore\quad f'(x)=e^{x\ln(x+1)}\left\{1.\ln(x+1)+\frac{x}{x+1}\right\}+e^{(x+1)\ln x}\left\{\frac{x+1}{x}+\ln x\right\}$

$=\ (x+1)^x\left\{\ln(x+1)+\frac{x}{x+1}\right\}+x^{x+1}\left\{\ln x+\frac{x+1}{x}\right\}$

$\therefore\ f'(1)=2\left(\ln 2+\frac{1}{2}\right)+1\{\ln 1+2\}$

$=\ 2\ln 2+1+2=2\ln 2+3$

3. $y=\frac{1+\sin\phi+\cos\phi}{1+\sin\phi-\cos\phi}$ হলে দেখাও যে $\frac{dy}{d\phi}+\frac{1}{1-\cos\phi}=0$

$$\therefore\ y=\frac{(1+\cos\phi)+\sin\phi}{(1-\cos\phi)+\sin\phi}=\frac{2\cos^2\frac{\phi}{2}+2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{2\sin^2\frac{\phi}{2}+2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}$$

$$=\frac{2\cos\frac{\phi}{2}\left(\cos\frac{\phi}{2}+\sin\frac{\phi}{2}\right)}{2\sin\frac{\phi}{2}\left(\sin\frac{\phi}{2}+\cos\frac{\phi}{2}\right)}$$

$$=\cot\frac{\phi}{2}$$

$$\therefore\ \frac{dy}{d\phi}=\frac{d}{d\phi}\left(\cot\frac{\phi}{2}\right)=-\cos ec^2\frac{\phi}{2}.\frac{1}{2}$$

$$=-\frac{1}{2\sin^2\frac{\phi}{2}}$$

$$=-\frac{1}{1-\cos\phi}$$

$$\therefore\ \frac{dy}{d\phi}+\frac{1}{1-\cos\phi}=0$$

4. $y\sqrt{x^2+1}=\log\left(\sqrt{x^2+1}-x\right)$ হলে দেখাও যে :

$$\left(x^2+1\right)\frac{dy}{dx}+xy+1=0$$

$$\therefore\ y\sqrt{x^2+1}=\log\left(\sqrt{x^2+1}-x\right)$$

$$\therefore\ \frac{dy}{dx}\sqrt{x^2+1}+y.\frac{2x}{2\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}-x}.\left(\frac{2x}{2\sqrt{x^2+1}}-1\right)$$

$$\therefore\frac{dy}{dx}\sqrt{x^2+1}+\frac{xy}{\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}-x}.\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

$$=-\frac{1}{\sqrt{x^2+1}}$$

$\therefore \ (x^2+1)\frac{dy}{dx}+xy=-1$

$\therefore \ (x^2+1)\frac{dy}{dx}+xy+1=0$

5. $x=a\cot\theta;\ y=\frac{1}{x^2+a^2}$ হলে দেখাও যে :

$\frac{d^2y}{dx^2}=\frac{2}{a^4}\sin^3\theta\sin 3\theta$

$\therefore \ y=\frac{1}{x^2+a^2}=\frac{1}{a^2\cos ec^2\theta}=\frac{1}{a^2}\sin^2\theta$

$\therefore \ \frac{dy}{d\theta}=\frac{2}{a^2}\sin\theta\cos\theta$

$\therefore \ x=a\cot\theta\Rightarrow\frac{dx}{d\theta}=-a\cos ec^2\theta$

$\therefore \ \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{2/a^2\sin\theta\cos\theta}{-a\cos ec^2\theta}=-\frac{2}{a^3}\sin^3\theta\cos\theta$

$\therefore \ \frac{d^2y}{dx^2}=-\frac{2}{a^3}.\frac{d}{dx}\left(\sin^3\theta\cos\theta\right)$

$=-\frac{2}{a^3}.\frac{d}{d\theta}\left(\sin^3\theta\cos\theta\right).\frac{d\theta}{dx}$

$=-\frac{2}{a^3}\left[3\sin^2\theta\cos^2\theta-\sin^4\theta\right]\frac{1}{-a\cos ec^2\theta}$

$=\frac{2}{a^4}\sin^3\theta\ \left[3\sin\theta(1-\sin^2\theta)-\sin^3\theta\right]$

$=\frac{2}{a^4}\sin^3\theta(3\sin\theta-4\sin^3\theta)$

$=\frac{2}{a^4}\sin^3\theta\sin 3\theta$

6. $\log x=z$ হলে দেখাও যে $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$

$\because \ \frac{dy}{dx}=\frac{dy}{dz}.\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}$

$\therefore \ \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}.\frac{dy}{dz}\right)$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}.\frac{d}{dz}\left(\frac{dy}{dz}\right).\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\therefore\ x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2}$$

7. $\sin x$ এর সাপেক্ষে $\tan x$ এর অবকল গুণাঙ্ক নির্ণয় কর :

ধরি, $y = \tan x;\ z = \sin x$

$$\frac{dy}{dx} = \sec^2 x;\ \ \frac{dz}{dx} = \cos x$$

$$\therefore\ \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\sec^2 x}{\cos x} = \sec^3 x$$

8. x-এর সাপেক্ষে অন্তরকলজ নির্ণয় করো : $(\sin x)^{\log x}$

$$y = (\sin x)^{\log x} \Rightarrow \log_e y = \log_e x \log_e \sin x$$

$$\therefore\ \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\log_e \sin x + \log_e x.\frac{1}{\sin x}\cos x$$

$$= \frac{1}{x}\log_e \sin x + \cot x \log_e x$$

$$\therefore\ \frac{dy}{dx} = (\sin x)^{\log x}\left[\frac{1}{x}\log \sin x + \cot x \log_e x\right]$$

9. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ হলে $\frac{d^2y}{dx^2}$ -এর মান নির্ণয় কর :

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\therefore \quad \frac{dy}{dx}=\frac{x}{a^2}.\frac{b^2}{y}$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{b^2}{a^2}.\frac{y.1-x\frac{dy}{dx}}{y^2}$$

$$= \frac{b^2}{a^2}.\frac{y-x.\frac{b^2x}{a^2y}}{y^2}$$

$$= \frac{b^2}{a^2}.\frac{a^2y^2-b^2x^2}{a^2y^3} \qquad \because \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow b^2x^2-a^2y^2=a^2b^2$$

$$\Rightarrow a^2y^2-b^2x^2=-a^2b^2$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{b^2}{a^2}\left(\frac{-a^2b^2}{a^2y^3}\right)$$

$$= -\frac{b^4}{a^2y^3}$$

10. $y=\sqrt{x+\sqrt{x+\sqrt{x+.....\infty}}}$ হলে $\frac{dy}{dx}$ -এর মান বাহির কর :

$$y=\sqrt{x+y} \Rightarrow y^2=x+y$$

$$\Rightarrow 2y\frac{dy}{dx}=1+\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(2y-1)=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2y-1}$$